Critical exponents for the restricted sandpile

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I report large-scale Monte Carlo studies of a one-dimensional height-restricted stochastic sandpile using the quasistationary simulation method. Results for systems of up to 50 000 sites yield estimates for critical exponents that differ significantly from those obtained using smaller systems, but are consistent with recent predictions derived from a Langevin equation for stochastic sandpiles [Ramasco *et al.*, Phys. Rev. E **69**, 045105(R) (2004)]. This suggests that apparent violations of universality in one-dimensional sandpiles are due to strong corrections to scaling and finite-size effects

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I. INTRODUCTION

Sandpile models are the prime example of self-organized criticality (SOC) [1,2], a control mechanism that forces a system with an absorbing-state phase transition to its critical point [3,4], leading to scale invariance in the apparent absence of parameters [5]. SOC in a slowly driven sandpile corresponds to an absorbing-state phase transition in a model having the same local dynamics, but a fixed number of particles [3,6–10]. The latter class of models is usually designated as *fixed-energy sandpiles* (FES) or *conserved sandpiles*. Continuous absorbing-state phase transitions characterized by a nonconserved order parameter (activity density) coupled to a conserved field that does not diffuse in the absence of activity, are expected to define a universality class [11]. This class, referred to as CDP (that is, a model-C version, in the sense of Halperin and Hohenberg [12], of directed percolation, or DP), appears to be distinct from that of directed percolation [13].

In recent years considerable progress has been made in characterizing the critical properties of conserved stochastic sandpiles, although no complete, reliable theory is yet at hand. As is often the case in critical phenomena, theoretical understanding of scaling and universality rests on the analysis of a continuum field theory or Langevin equation (a nonlinear stochastic partial differential equation) that reproduces the phase diagram and captures the fundamental symmetries and conservation laws of the system. Important steps in this direction are the recent numerical studies of a Langevin equation [13,14] for CDP. (The latter appears to incorporate the essential aspects of stochastic sandpiles.) The critical exponent values reported in Ref. [13] are in good agreement with simulations of conserved lattice gas (CLG) models [19,20], which exhibit the same symmetries and conservation laws as stochastic sandpiles.

The Langevin equation exponents are also consistent with the best available estimates for stochastic sandpiles in two dimensions [13], with the exception of the exponent θ governing the initial decay of the order parameter. (The discrepancy regarding θ likely reflects strong corrections to short-time scaling in sandpiles, due to long memory effects associated with initial density fluctuations [15].) Pending a better understanding of this question, it appears that stochastic sandpiles are consistent with CDP in two dimensions. In the one-dimensional case, however, there is a significant discrepancy between the Langevin equation results and those for sandpile models.

Specifically, analysis of the Langevin equation for CDP yields, in one dimension, the order-parameter critical exponent value $\beta=0.28(2)$, while previous studies [15–18] of stochastic sandpiles furnish values near 0.40 for this exponent. There are also smaller discrepancies for other critical exponents. If this discrepancy were to persist, one would be forced to conclude that the proposed Langevin equation misses some essential aspect of sandpiles (at least in the one-dimensional case), or that not all models with the same symmetries and conserved quantities belong to the same universality class. In an effort to clarify the situation, I apply the recently devised quasistationary simulation method [21–23] to the restricted-height sandpile introduced in Ref. [16].

The balance of this paper is organized as follows. In Sec. II we define the model and summarize the simulation method. Numerical results are analyzed in Sec. III, and Sec. IV discusses the findings in the context of universality.

II. MODEL

I study the "independent" version of the model introduced in Ref. [16]. The system, a continuous-time, restricted-height version of Manna's stochastic sandpile [24], is defined on a ring of L sites. The configuration is specified by the number of particles, $z_i=0,1$, or 2, at each site i. Sites with $z_i=2$ are *active*, and have a toppling rate of unity. The continuous-time Markovian dynamics consists of a series of toppling events at individual sites. When site i topples, two particles attempt to move randomly (and independently) to either $i-1$ or $i+1$. (The two particles may both try to jump to the same neighbor.) Each particle transfer is accepted so long as it does not lead to a site having more than two particles. (If the target site is already doubly occupied the particle does not move. Thus an attempt to send two particles from site j

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to site k, with $z_k=1$, results in $z_k=2$ and $z_j=1$.) The next site to topple is chosen at random from a list of active sites, which is updated following each event. The time increment associated with each toppling is $\Delta t=1/N_A$, where N_A is the number of active sites just prior to the event.

Any configuration devoid of doubly occupied sites is absorbing. Although absorbing configurations exist for particle densities $p=N/L\leq 1$, the critical value p_c (above which activity continues indefinitely) appears to be strictly less than unity. In Ref. [16] the model was studied in the site and pair mean-field approximation (which yield a continuous phase transition at $p_c=0.5$ and 0.75, respectively, in one dimension), and via Monte Carlo simulation using system sizes of up to 5000 sites. The latter yield the estimates $p_c=0.929\,65(3)$, $\beta/\nu_\perp=0.247(2)$, $z=\nu_\|/\nu_\perp=1.45(3)$, and $\beta=0.412(4)$. A similar value, $\beta=0.42(1)$, was obtained in Ref. [17] using a series of cluster approximations (of up to 11 sites), combined with Suzuki's coherent anomaly analysis [25].

The studies reported here employ the quasistationary (QS) simulation method, which, due to increased efficiency in the critical region, permits a tenfold increase in the system size as compared to Ref. [16]. The QS method, described in detail in Ref. [21], provides a just sampling of asymptotic (long-time) properties, conditioned on survival. In practice this is accomplished by maintaining (and gradually updating) a set of configurations visited during the evolution; when a transition to the absorbing state is imminent the system is instead placed in one of the saved configurations. Otherwise the evolution is exactly that of a "standard" simulation algorithm such as used in Ref. [16].

III. SIMULATION RESULTS

I performed two sets of studies using the QS method. The first is used to determine the QS order parameter (defined as the faction ρ of active sites), the moment ratio $m=\langle\rho^2\rangle/\rho^2$, and the mean lifetime τ of the quasistationary state, in the immediate vicinity of the critical point p_c, for system sizes $L=1000$, 2000, 5000, 10 000, 20 000, and 50 000. (The QS lifetime is taken as the mean number of time steps between successive attempts to visit the absorbing state.) A second set of simulations is used to study the supercritical regime ($p>p_c$) for system sizes $L=10\,000$, 20 000, and 50 000. (For p substantially larger than p_c, the lifetime is much larger than the simulation time, so that the system never visits the absorbing state, and the QS method becomes identical to a standard simulation.)

Each realization of the process is run for 10^9 time steps; averages are taken in the QS regime, which necessitates discarding an initial transient that ranges from 10^6 time steps (for $L=1000$) to 10^8 time steps (for $L=50\,000$). The number of saved configurations ranges from 1000 (for $L=1000$) to 400 (for $L=50\,000$). The list updating probability $p_{\rm rep}$ ranges from 10^{-3} (for $L=1000$) to 5×10^{-6} (for $L=50\,000$). During the initial relaxation period $p_{\rm rep}$ is increased by a factor of 10 to erase the memory of the initial configuration.

I first discuss the studies focusing on the critical region. As in Ref. [16], I study for each system size, a series of particle number values N, chosen so that $p=N/L$ lies immediately above or below p_c. Since the particle density can only be varied in steps of $1/L$, estimates for properties at intermediate values of p are obtained via interpolation. The results of the QS simulations were found to agree, to within uncertainty, with the corresponding results of conventional simulations [16], for $L=1000$, 2000, and 5000. The criterion for criticality is power-law dependence of ρ and τ on system size, i.e., the familiar relations $\rho\sim L^{-\beta/\nu_\perp}$ and $\tau\sim L^z$, and constancy of the moment ratio m with L. The most sensitive indicator turns out to be the order parameter ρ. Using the data for system sizes 5000–50 000, I rule out p values that yield a statistically significant curvature of the graph of $\ln\rho$ versus $\ln L$. This results in the estimate $p_c=0.929\,780(7)$. (For the remainder of the analysis p_c is fixed at this value and is no longer available as an adjustable parameter.) The associated exponent is $\beta/\nu_\perp=0.213(6)$, where the uncertainty represents a contribution (±0.005) due to the uncertainty in p_c and a small additional uncertainty in the linear fit to the data. Simulation results for ρ as a function of L, for various densities near p_c, are shown in Fig. 1; curvature of the plots for off-critical values is evident in the inset.

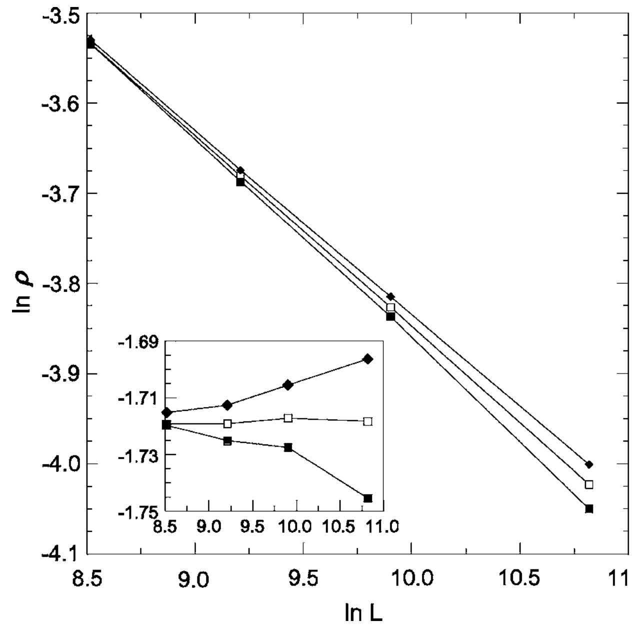

FIG. 1. Stationary order parameter versus system size for particle densities (bottom to top) $p=0.929\,77$, 0.929 78, and 0.929 79. Inset, $\ln L^{0.213}\rho$ versus $\ln L$ for the same set of particle densities.

The data for the QS lifetime τ furnish a similar but somewhat less precise estimate, $p_c=0.929\,777(17)$. Fitting the data for $L=5000-50\,000$, using the p_c interval obtained from the analysis of ρ, I find $z=1.50(4)$. The moment ratio m is also useful for setting limits on p_c. As shown in Fig. 2, this quantity appears to grow with system size for $p<p_c$ and vice versa; we may rule out the values 0.929 76 and 0.929 80 on this basis. The moment ratio data yield $m_c=1.142(8)$. The main contribution to the uncertainties in z and m is again due to the uncertainty in p_c.

The present estimate for p_c is significantly greater than that found in Ref. [16], although the difference amounts to

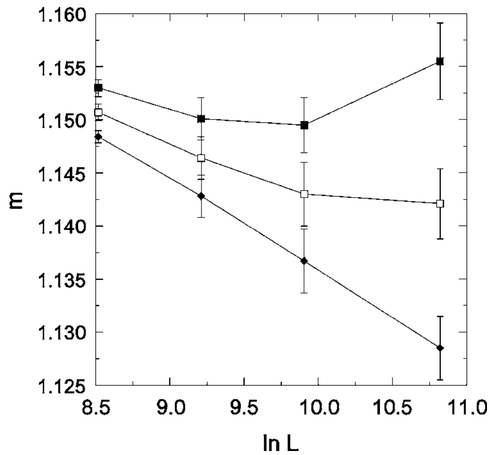

FIG. 2. Moment ratio m versus system size for particle densities (top to bottom) $p=0.929\,76$, $0.929\,78$, and $0.929\,80$.

about 0.01%. The results for the exponent z are consistent, but the present study yields a substantially (16%) lower estimate for $\beta/\nu_\perp$ than reported previously. The present result for m_c is also substantially lower than the value 1.1596(4) reported in Ref. [16]. These differences highlight the strong finite-size corrections affecting stochastic sandpiles.

Next the results for the order parameter in the supercritical regime are analyzed. Figure 3 shows that the data for system sizes 10 000, 20 000, and 50 000 are well converged for $\Delta=p-p_c\geq 10^{-3}$, that is, finite-size effects are only present nearer the critical point. Evidently, the data are not consistent with a simple power law of the form $\rho\sim\Delta^\beta$. Indeed this departure from the familiar behavior of the order parameter was already noted (with data for smaller systems) in Ref. [16]. In the latter work the power law was "restored" by introducing a size-dependent critical density $p_c(L)\simeq p_{c,\infty}-\text{const}/L^{1/\nu_\perp}$, leading to a series of estimates for the critical exponent β that increase systematically with L, apparently converging to $\beta=0.412(4)$. With the present data, which are converged over a broader range of Δ values, I find that shifting the critical value *does not lead to an apparent power law*.

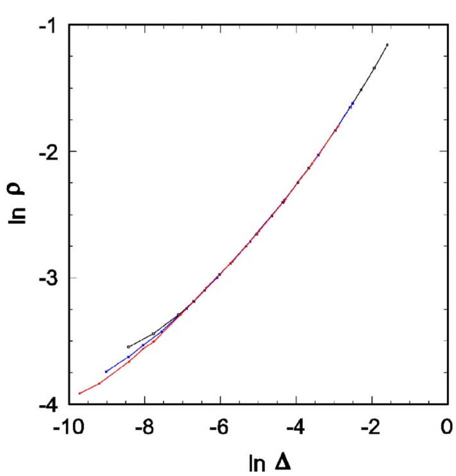

FIG. 3. (Color online) Stationary order parameter versus $\Delta=p-p_c$ for system sizes (top to bottom) $L=10^4$, 2×10^4, and 5×10^4.

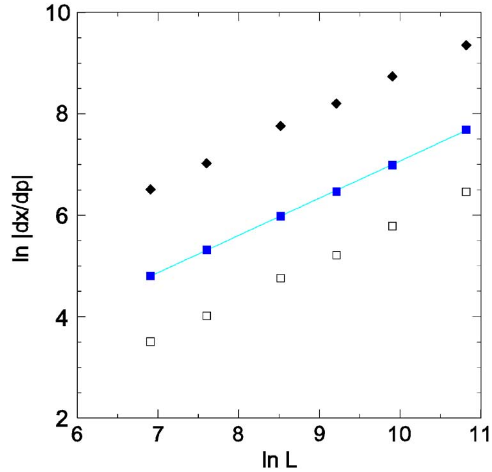

FIG. 4. (Color online) Derivatives of (lower to upper) ln τ, ln ρ, and m with respect to particle density, evaluated at p_c, versus system size. The slope of the straight line is 0.734.

One is therefore left to conclude that either the order parameter does not obey power-law scaling, or that there are unusually strong corrections to scaling. Including a correction to scaling term, one has

$$\rho\sim\Delta^\beta(1+A\Delta^{\beta'}) \tag{1}$$

so that there are now three adjustable parameters, β, β', and A. Even with a reasonably large number of data points (18 for $L=10\,000$), this induces a huge range of variation in the exponent β. Decent fits can be obtained with values as low as $\beta=0.1$ and as large as 0.3.

To resolve this difficulty I return to the data in the immediate vicinity of p_c. These data can be used to determine the correlation length exponent $\nu_\perp$ in the following manner. Finite-size scaling implies that for $p\simeq p_c$, the moment ratio obeys the relation

$$m(\Delta,L)\simeq\mathcal{F}_m(L^{1/\nu_\perp}\Delta), \tag{2}$$

where $\mathcal{F}_m$ is a scaling function. This implies that

$$\left|\frac{\partial m}{\partial p}\right|_{p_c}\propto L^{1/\nu_\perp}. \tag{3}$$

Moreover, the finite-size expression $\rho=L^{-\beta/\nu_\perp}\mathcal{F}_\rho(L^{1/\nu_\perp}\Delta)$ implies that

$$\left|\frac{\partial\ln\rho}{\partial p}\right|_{p_c}\propto L^{1/\nu_\perp}, \tag{4}$$

and similarly for the derivative of ln τ at the critical point. The derivatives are evaluated numerically as follows. For each value of L studied, data for five values of p clustered around p_c are fit with a cubic polynomial; the derivative of the polynomial is then evaluated at p_c. The resulting derivatives are plotted in Fig. 4; clean power laws are observed, leading to $\nu_\perp=1.362(7)$, 1.323(14), and 1.372(21), using the data for ln ρ, m, and ln τ, respectively. Pooling these results yields the estimate $\nu_\perp=1.355(18)$. Then, using the values for

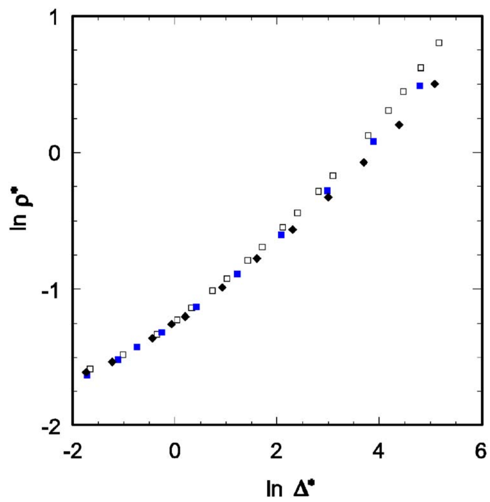

FIG. 5. (Color online) Scaled density ρ^* versus scaled distance from critical point Δ^*, as defined in text. System sizes, 10^4 (open squares); 2×10^4 (filled squares); 5×10^4 (diamonds).

$\beta/\nu_\perp$ and z reported above, I find $\beta=0.289(12)$ and $\nu_\parallel=2.03(8)$.

Using this value for β, the data for the order parameter in the supercritical regime can be fit using the correction to scaling form, Eq. (1), with parameters $\beta'=0.446$ and $A=1.3505$. For $\Delta=0.1$, the correction term $A\Delta^{\beta'}$ in Eq. (1) is 0.48, showing that there are sizeable deviations from a pure power law. It is usual to verify scaling by seeking a data collapse, plotting $\rho^*=L^{\beta/\nu_\perp}\rho$ versus $\Delta^*=L^{1/\nu_\perp}\Delta$. For $\Delta>0.001$ the order parameter does not follow a pure power law and so the data cannot collapse. It is nevertheless of interest to construct such a scaling plot (Fig. 5). Although the data do not collapse over most of the range, they do collapse in the interval $-1\leqslant\Delta\leqslant1$. A linear fit to the data in this interval yields a slope of 0.27(1). This is close to the β value obtained from the finite-size scaling analysis, suggesting that simple scaling is restricted to a narrow interval very near the critical point.

IV. DISCUSSION

A study of the one-dimensional restricted-height stochastic sandpile using quasistationary simulations permits study of systems an order of magnitude larger than previously studied, and yields critical properties different from those obtained previously. In the case of the critical density, the small change (about 0.01%) from the previous estimate may be attributed to finite-size effects, which are known to affect sandpile models strongly.

Of greater concern are critical exponent values, since they define the universality class of the model. Since there is every reason (based on symmetry considerations) to expect the restricted sandpile to belong to the same universality class as the unrestricted version (indeed, this seems well established in two dimensions [16]), Table I compares critical exponent values from various studies of stochastic sandpiles, CDP, and the conserved threshold transfer process (CTTP), also expected to belong to the same class.

The overall conclusion from Table I is that studies using smaller lattices yield values in the range 0.38–0.42 for the exponent β (Ref. [18] is however an exception), and that the large-scale simulation of Ref. [20], the numerical study of the CDP field theory [13] and the present work yield a consistent set of results, with $\beta\simeq0.29$. (A similar value has been found for a modified conserved lattice gas model [27].) Although the system size (4000 sites) used in the field theory simulations is not large, one should note that each site in such a simulation may represent a region comprising many lattice sites in the original model. Compared with the earlier sandpile simulations, the distinctive feature of the present work may not be system size, but the fact that here the exponent β is determined via finite-size scaling *at the critical point*, rather than from the usual analysis of the order parameter in the supercritical regime. Indeed, it is easy to see from Fig. 3 that data for $\Delta=p-p_c$ in the range 10^{-3}–10^{-1} will yield larger estimates for β. [The same observation applies to the coherent anomaly method (CAM) analysis [17], which essentially probes the shape of the function $\rho(\Delta)$ at some distance from the critical point $\Delta=0$.] I observe a simple power-law behavior, and data collapse for various lattice sizes, only in a restricted range of the scaling variable $\Delta^*=L^{1/\nu_\perp}\Delta$.

Also included in Table I are exponent values for one-dimensional directed percolation [28]. The values obtained in Refs. [13,20], as well as in the present work, are not very different from those of DP. A clear difference from DP scaling was however demonstrated in Ref. [14], where the initial

TABLE I. Summary of exponent values for one-dimensional models in the CDP universality class. L_{max} denotes the largest system size studied. Abbreviations: CAM, coherent anomaly method; FT, field theory.

Model	L_{max}	β	$\beta/\nu_\perp$	z
Manna [15]	10000	0.42(2)	0.24(1)	1.66(7)
Manna [26]	8192		0.28(3)	1.39(11)
CTTP [18]	131072	0.38(2)	0.24(1)	1.66(7)
Rest. Manna [16]	5000	0.416(4)	0.246(5)	1.50(9)
Rest. Manna CAM [17]		0.41(1)		
CDP [20]	4.2×10^6	0.29(2)		1.55(3)
CDP FT [13]	4000	0.28(2)	0.214(8)	1.47(4)
Rest. Manna (present work)	50000	0.289(12)	0.213(6)	1.50(4)
DP [28]		0.2765	0.2521	1.5807

decay exponent for one-dimensional CDP is found to be $\theta=0.125(2)$, as opposed to 0.1595(1) for DP. The rather substantial differences found here in $\beta/\nu_{\perp}$, and in the moment ratio m [1.142(8) for the restricted sandpile compared with 1.1736(1) for DP [21]], lend further support to the conclusion that the CDP/stochastic sandpile universality class is distinct from that of directed percolation, as is evidently the case in two dimensions. (This is despite the result in Ref. [29], that when suitably modified to include sticky grains, sandpiles fall generically in the DP class.)

In summary, I have applied the quasistationary simulation method to a one-dimensional restricted-height stochastic sandpile, and the obtained results are consistent with recent studies of CDP. This supports the assertion that the latter class includes stochastic sandpiles, as would be expected on the basis of symmetry and conservation laws.

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